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Solution of the Dirac equation on the homogeneous manifold $SL(2,c)/GL(1,c)$ in the presence of a magnetic monopole field

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Abstract. Equations of shape invariance have been derived on the homogeneous manifold $SL(2,c)/GL(1,c)$, by means of which the Dirac equation is solved for a charged spin-$\frac{1}{2}$ particle in the presence of a magnetic monopole. The Dirac spinors on this manifold are written in terms of the master function. It is shown that these spinors represent an $N = 1$ chiral supersymmetry algebra and a unitary parasuperalgebra of arbitrary order p.

Over the last few years, the ideas of supersymmetry [1] and shape invariance [2, 3] have been successfully applied to many simple quantum mechanical systems. Supersymmetry is a symmetry between bosonic and fermionic degrees of freedom. Supersymmetry transformations are connected Hamiltonians of two systems in supersymmetric quantum mechanics. These two Hamiltonians have, except for the ground state, the same spectra. Later, the above significant concept for supersymmetric quantum mechanics was extended to the concept of shape invariance. The Hamiltonians satisfy the shape invariance condition, and the corresponding spectra and wavefunctions are exactly determined by elementary calculations and algebraic procedures. In [3] a selection of special functions were obtained as solutions of the Schrödinger equation with shape invariance potentials. From the existence of shape invariance symmetry, supersymmetry extends to parasupersymmetry [4–6] which describes symmetry between bosons and parafermions. In usual supersymmetric quantum mechanics, the symmetry generators obey structure relations that involve bilinear products, whereas in parasupersymmetric quantum mechanics of order p, the structure relations involve products of $(p + 1)$ parasupersymmetry charges. It is pointed out that while in supersymmetric quantum mechanics the energy eigenvalues are necessarily non-negative, in parasupersymmetric quantum mechanics of order p they need not be so. These important arguments have begun with quantum mechanical problems in one-dimensional space. Of course there have been some attempts to solve the Schrödinger equation for two-dimensional potentials from the shape invariance approach [7–9].

In this paper, using the ideas of supersymmetric quantum mechanics and shape invariance we obtain a solution of the Dirac equation for a charged particle with spin $\frac{1}{2}$ on the homogeneous manifold $SL(2,c)/GL(1,c)$ in the presence of a magnetic monopole field. In fact, one can infer that supersymmetry, shape invariance and parasupersymmetry are represented by Dirac spinors on homogeneous manifolds $SU(2)/U(1)$, $SU(1,1)/U(1)$ and $H_4/(U(1)\otimes U(1))$ as real forms of the $SL(2,c)/GL(1,c)$ manifold.

The paper is arranged as follows. In section 2 operators of the $gl(2,c)$ Lie algebra on the group manifold $SL(2,c)$ have been derived in terms of the master function and a weight function. Then, by reducing the parameter $GL(1,c)$ from a representation of the $gl(2,c)$ Lie algebra, we deduce the operators which describe the shape invariance on the homogeneous manifold $SL(2,c)/GL(1,c)$. Also, we introduce the metric of this manifold in terms of the master function. In section 3, it is shown that shape invariance obtained on a $SL(2,c)/GL(1,c)$ manifold culminates in the solution of the Dirac equation on this manifold for a charged particle in the presence of a magnetic monopole. In section 4 it has been shown that Dirac spinors on a $SL(2,c)/GL(1,c)$ manifold represent an $N=1$ chiral supersymmetry algebra. In section 5, by using the shape invariance obtained in section 3, we show that spinors realize a shape invariance as if they were obtained by an algebraic procedure. Then, by means of some suitable parafermionic and bosonic generators, we consider a representation of the parasuperalgebra of arbitrary order p by Dirac spinors on the homogeneous manifold $SL(2,c)/GL(1,c)$.

2. Shape invariance symmetry on the homogeneous manifold $SL(2,c)/GL(1,c)$

In [3, 8], for a given master function $A(x)$ which is a polynomial of at most degree two, the weight function $W(x)$ and the interval $[a,b]$ are allocated such that $A(x)W(x)$ and all of its derivatives are zero at the end points. The associated differential equations of mathematical physics have been introduced in terms of the master function $A(x)$ as follows:

$$A(x)\Phi''_{n,m}(x)+\frac{(A(x)W(x))'}{W(x)}\Phi'_{n,m}(x)+\left[-\frac{1}{2}(n^2+n-m^2)A''(x)+(m-n)\left(\frac{A(x)W'(x)}{W(x)}\right)'\right.$$
$$\left.-\frac{m^2}{4}\frac{(A'(x))^2}{A(x)}-\frac{m}{2}\frac{A'(x)W'(x)}{W(x)}\right]\Phi_{n,m}(x)=0 \qquad m=0,1,2,\ldots,n \tag{2.1}$$

where the associated special functions are introduced as its solutions:

$$\Phi_{n,m}(x)=\frac{a_n(-1)^m}{A^{m/2}(x)W(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}(A^n(x)W(x)). \tag{2.2}$$

These special functions depend on two indices, one of which is the main index n and the other is the associated index m, where n is a natural number and $m=0,1,\ldots,n$. With the change of variable as

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sqrt{A(x)} \tag{2.3}$$

and using equation (2.1) we obtain the following shape invariance equations [3]:

$$\begin{aligned}B(m)A(m)\,\psi_{n,m}(\theta)&=E(n,m)\,\psi_{n,m}(\theta)\\ A(m)B(m)\,\psi_{n,m-1}(\theta)&=E(n,m)\,\psi_{n,m-1}(\theta)\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}B(m)&=\frac{\mathrm{d}}{\mathrm{d}\theta}+W_m(\theta)\\ A(m)&=-\frac{\mathrm{d}}{\mathrm{d}\theta}+W_m(\theta)\end{aligned} \tag{2.5}$$

and

$$\psi_{n,m}(\theta) = a_n(-1)^m \left\{ A^{(-2m+1)/4}(x) W^{-1/2}(x) \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m} \left(A^n(x)W(x)\right)\right\}_{x=x(\theta)}. \tag{2.6}$$

The superpotential $W_m(\theta)$ and the spectrum $E(n,m)$ have already been introduced as the following equations:

$$W_m(\theta) = -\left.\frac{\frac{1}{2}(A(x)W'(x)/W(x)) + \frac{1}{4}(2m-1)A'(x)}{\sqrt{A(x)}}\right|_{x=x(\theta)} \tag{2.7}$$

$$E(n,m) = -(n-m+1)\left[\left(\frac{A(x)W'(x)}{W(x)}\right)' + \frac{1}{2}(n+m)A''(x)\right]. \tag{2.8}$$

So studying shape invariance on m resulted in introducing a bunch of one-dimensional superpotentials (2.7). This dimension is due to the space coordinate θ, that was naturally obtained from the master function $A(x)$ by a change of variable (2.3).

From now on we shall only consider the case in which $A'^2(0) - 2A''A(0) > 0$.

In order to introduce the operators of the $gl(2,c)$ Lie algebra on the group manifold $SL(2,c)$, we define a new parameter l as

$$l := -\frac{1}{2}\left(\frac{AW'}{W}\right)(0). \tag{2.9}$$

We present appropriate variables ψ and ϕ corresponding to m and l, respectively, as follows: in the raising and lowering operators $B(m)$ and $A(m)$ we replace $-\mathrm{i}\partial/\partial\psi$ for $m-1$ and m, respectively, and also substitute $-\frac{1}{2}\mathrm{i}\sqrt{A'^2(0) - 2A''A(0)}\,\partial/\partial\phi$ for l in both operators to obtain the following generators:

$$\begin{aligned}
J_+ &= \mathrm{e}^{\mathrm{i}\psi}\left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{2}\sqrt{\frac{A'^2(0)-2A''A(0)}{A(x)}}\frac{\partial}{\partial\phi} + \frac{\mathrm{i}A'(x)}{2\sqrt{A(x)}}\frac{\partial}{\partial\psi}\right.\\
&\qquad \left. - \frac{1}{2\sqrt{A(x)}}\left(\frac{AW'}{W}\right)'x - \frac{A'(x)}{4\sqrt{A(x)}}\right)\\
J_- &= \mathrm{e}^{-\mathrm{i}\psi}\left(-\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{2}\sqrt{\frac{A'^2(0)-2A''A(0)}{A(x)}}\frac{\partial}{\partial\phi} + \frac{\mathrm{i}A'(x)}{2\sqrt{A(x)}}\frac{\partial}{\partial\psi}\right.\\
&\qquad \left. - \frac{1}{2\sqrt{A(x)}}\left(\frac{AW'}{W}\right)'x + \frac{A'(x)}{4\sqrt{A(x)}}\right).
\end{aligned} \tag{2.10}$$

Here, θ, ϕ and ψ are an appropriate parametrization for the group manifold $SL(2,c)$ [9]. These operators together with

$$J_3 = -\mathrm{i}\frac{\partial}{\partial\psi} \qquad I = 1 \tag{2.11}$$

satisfy the $gl(2,c)$ Lie algebra

$$\begin{aligned}
&[J_+, J_-] = -A''(x)J_3 - \left(\frac{AW'}{W}\right)' I\\
&[J_3, J_\pm] = \pm J_\pm\\
&[\boldsymbol{J}, I] = 0.
\end{aligned} \tag{2.12}$$

Using the shape invariance (2.4), the representation of the $gl(2,c)$ Lie algebra on the group manifold $SL(2,c)$ is given by

$$\begin{aligned}
J_+\psi_{n,l,m}(\theta,\phi,\psi) &= \sqrt{E(n,m+1)}\,\psi_{n,l,m+1}(\theta,\phi,\psi)\\
J_-\psi_{n,l,m}(\theta,\phi,\psi) &= \sqrt{E(n,m)}\,\psi_{n,l,m-1}(\theta,\phi,\psi)\\
J_3\psi_{n,l,m}(\theta,\phi,\psi) &= m\psi_{n,l,m}(\theta,\phi,\psi)\\
I\psi_{n,l,m}(\theta,\phi,\psi) &= \psi_{n,l,m}(\theta,\phi,\psi)
\end{aligned} \tag{2.13}$$

where the bases of representation are

$$\psi_{n,l,m}(\theta,\phi,\psi) = \exp\left(\frac{2\mathrm{i}l}{\sqrt{A'^2(0)-2A''A(0)}}\phi + \mathrm{i}m\psi\right)\psi_{n,m}(\theta). \tag{2.14}$$

The bases (2.14) also depend on parameter l through the weight function $W(x)$. Indeed, for functions $\psi_{n,l,m}(\theta,\phi,\psi)$ the shape invariance (2.4) is written as

$$\begin{aligned}
J_+J_-\psi_{n,l,m}(\theta,\phi,\psi) &= E(n,m)\,\psi_{n,l,m}(\theta,\phi,\psi)\\
J_-J_+\psi_{n,l,m-1}(\theta,\phi,\psi) &= E(n,m)\,\psi_{n,l,m-1}(\theta,\phi,\psi).
\end{aligned} \tag{2.15}$$

Since either ϕ or ψ can be considered as the only parameter of $GL(1,c)$, we reduce one of them, so that the remaining parameters, i.e. either $\{\theta,\phi\}$ or $\{\theta,\psi\}$, can be considered as the parameters that describe the homogeneous manifold $SL(2,c)/GL(1,c)$ [9, 10]. If ψ is reduced from the $SL(2,c)$ manifold, then from the first two equations of (2.13) we obtain the raising and lowering operators of states on the homogeneous manifold $SL(2,c)/GL(1,c)$:

$$\begin{aligned}
J_+(m) &= \frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{2}\sqrt{\frac{A'^2(0)-2A''A(0)}{A(x)}}\frac{\partial}{\partial\phi} - \frac{mA'(x)}{2\sqrt{A(x)}}\\
&\quad - \frac{1}{2\sqrt{A(x)}}\left(\frac{AW'}{W}\right)'x - \frac{A'(x)}{4\sqrt{A(x)}}\\
J_-(m) &= -\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{2}\sqrt{\frac{A'^2(0)-2A''A(0)}{A(x)}}\frac{\partial}{\partial\phi}\\
&\quad - \frac{mA'(x)}{2\sqrt{A(x)}} - \frac{1}{2\sqrt{A(x)}}\left(\frac{AW'}{W}\right)'x + \frac{A'(x)}{4\sqrt{A(x)}}
\end{aligned} \tag{2.16}$$

such that

$$\begin{aligned}
J_+(m)\,\psi_{n,l,m-1}(\theta,\phi,0) &= \sqrt{E(n,m)}\,\psi_{n,l,m}(\theta,\phi,0)\\
J_-(m)\,\psi_{n,l,m}(\theta,\phi,0) &= \sqrt{E(n,m)}\,\psi_{n,l,m-1}(\theta,\phi,0).
\end{aligned} \tag{2.17}$$

In other words, with the help of equations (2.17) (or directly by reducing ψ in equations (2.15)), the shape invariance equations on the homogeneous manifold $SL(2,c)/GL(1,c)$ are obtained as

$$\begin{aligned}
J_+(m)J_-(m)\,\psi_{n,l,m}(\theta,\phi,0) &= E(n,m)\,\psi_{n,l,m}(\theta,\phi,0)\\
J_-(m)J_+(m)\,\psi_{n,l,m-1}(\theta,\phi,0) &= E(n,m)\,\psi_{n,l,m-1}(\theta,\phi,0).
\end{aligned} \tag{2.18}$$

Thus, operators $J_+(m)$ and $J_-(m)$ describe the shape invariance symmetry on the homogeneous manifold $SL(2,c)/GL(1,c)$ parametrized by $\{\theta,\phi\}$.

As in [8, 9], we would like to point out that by this kind of parametrizing $SL(2,c)/GL(1,c)$ manifold and with the change of variable given in (2.3), we obtain the metric in terms of the master function as

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{4A(x)}{A'^2(0) - 2A''A(0)} \end{pmatrix} \tag{2.19}$$

where i and j denote θ and ϕ. The Ricci scalar curvature of the metric (2.19) for the homogeneous manifold $SL(2,c)/GL(1,c)$ is a constant: $R = -A''(x)$. In [9] for the metric (2.19) we have derived some quantum solvable models which correspond to the Schrödinger equation with the degeneracy group $GL(2,c)$ and without the degeneracy group. Here, for this metric, we consider the Dirac equation in Minkowskian spacetime; the space part of which is the $SL(2,c)/GL(1,c)$ manifold and we give Dirac solvable models on the $SL(2,c)/GL(1,c)$ manifold in the presence of the magnetic field of a monopole such that spinors of the Dirac equation are written in terms of the master function.

3. Master function approach to solution of the Dirac equation on the homogeneous manifold $SL(2,c)/GL(1,c)$

By using equation (2.19), the Minkowskian spacetime metric for the $SL(2,c)/GL(1,c)$ manifold is introduced as follows [9]:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \dfrac{-4A(x)}{A'^2(0) - 2A''A(0)} \end{pmatrix} \tag{3.1}$$

where μ and ν numerate rows and columns by t, θ and ϕ. The Dirac equation for the spacetime described by (3.1) is

$$D_{1+2}\Psi(t;\theta,\phi) = 0 \tag{3.2}$$

in which the Dirac operator D_{1+2} is [11]

$$D_{1+2} = -\mathrm{i}\gamma^a E_a{}^\mu\left(\partial_\mu - \mathrm{i}A_\mu + \tfrac{1}{8}\omega_{\mu\ ab}[\gamma^a,\gamma^b]\right). \tag{3.3}$$

The generators of Clifford algebra, i.e. γ^a, are defined as

$$\gamma^0 = \sigma^3 \qquad \gamma^1 = \mathrm{i}\sigma^2 \qquad \gamma^2 = -\mathrm{i}\sigma^1 \tag{3.4}$$

where σ^1, σ^2 and σ^3 are the known Pauli matrices. With regard to $\gamma_c := \eta_{cd}\gamma^d$, it is easy to show that for the Minkowskian diagonal metric $\eta^{ab} = (1,-1,-1)$, the generators γ^a satisfy

$$\gamma^a\gamma^b = \eta^{ab}I_{2\times 2} - \mathrm{i}\epsilon^{abc}\gamma_c. \tag{3.5}$$

The 3-bein $E_a{}^\mu$ and its inverse, i.e. $e_\mu{}^a$, for the spacetime metric (3.1) must satisfy the following relations:

$$\begin{aligned} E_a{}^\mu\eta^{ab}E_b{}^\nu &= g^{\mu\nu} \\ E_a{}^\mu g_{\mu\nu}E_b{}^\nu &= \eta_{ab} \end{aligned} \tag{3.6a}$$

$$\begin{aligned} e_\mu{}^a g^{\mu\nu}e_\nu{}^b &= \eta^{ab} \\ e_\mu{}^a\eta_{ab}e_\nu{}^b &= g_{\mu\nu}. \end{aligned} \tag{3.6b}$$

Also, the spin connection $\omega_{\mu\ ab}$ is defined by the following equation:

$$\partial_\mu e_\nu{}^a - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \tag{3.7}$$

in which $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols for the metric (3.1), the only non-vanishing components of which are

$$\Gamma^\theta_{\phi\phi} = \frac{-2A'(x)\sqrt{A(x)}}{A'^2(0) - 2A''A(0)} \qquad \Gamma^\phi_{\theta\phi} = \frac{A'(x)}{2\sqrt{A(x)}}. \tag{3.8}$$

From equations (3.6a) and (3.6b) we obtain the following for the metric (3.1) respectively,

$$E_a{}^\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \dfrac{\sqrt{A'^2(0) - 2A''A(0)}}{2\sqrt{A(x)}} \end{pmatrix}$$
$$e_\mu{}^a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \dfrac{2\sqrt{A(x)}}{\sqrt{A'^2(0) - 2A''A(0)}} \end{pmatrix}. \tag{3.9}$$

From equation (3.7) we calculate the non-vanishing components of the spin connection

$$\omega_\phi{}^1{}_2 = -\omega_\phi{}^2{}_1 = \frac{A'(x)}{\sqrt{A'^2(0) - 2A''A(0)}}. \tag{3.10}$$

Using these results together with equation (3.5), one can introduce the time-dependent Dirac equation (3.2) as follows:

$$\begin{pmatrix} -\partial_t + \mathrm{i}A_t & \partial_\theta + \dfrac{\mathrm{i}}{2}\sqrt{\dfrac{A'^2(0) - 2A''A(0)}{A(x)}}\partial_\phi - \mathrm{i}A_\theta + \dfrac{1}{2}\sqrt{\dfrac{A'^2(0) - 2A''A(0)}{A(x)}}A_\phi + \dfrac{A'(x)}{4\sqrt{A(x)}} \\ -\partial_\theta + \dfrac{\mathrm{i}}{2}\sqrt{\dfrac{A'^2(0) - 2A''A(0)}{A(x)}}\partial_\phi + \mathrm{i}A_\theta + \dfrac{1}{2}\sqrt{\dfrac{A'^2(0) - 2A''A(0)}{A(x)}}A_\phi - \dfrac{A'(x)}{4\sqrt{A(x)}} & \partial_t - \mathrm{i}A_t \end{pmatrix}$$
$$\times \Psi(t;\theta,\phi) = 0. \tag{3.11}$$

If we assume time evolution spinors as $\mathrm{e}^{-\mathrm{i}\sqrt{E(n,m)}t}$ and compare the Dirac equation (3.11) with equations (2.17), the spinors of a charged spin-$\frac{1}{2}$ particle become

$$\Psi_{n,l,m}(t;\theta,\phi) = \mathrm{e}^{-\mathrm{i}\sqrt{E(n,m)}t}\Psi_{n,l,m}(\theta,\phi)$$
$$= \mathrm{e}^{-\mathrm{i}\sqrt{E(n,m)}t}\begin{pmatrix} \psi_{n,l,m-1}(\theta,\phi,0) \\ \mathrm{i}\psi_{n,l,m}(\theta,\phi,0) \end{pmatrix} \qquad m = 0, 1, \ldots, n+1 \tag{3.12}$$

where we define $\psi_{n,l,-1}(\theta,\phi,0) = \psi_{n,l,n+1}(\theta,\phi,0) = 0$. From this comparison we find that

$$\begin{aligned} A_t &= 0 \\ A_\theta &= \frac{-\mathrm{i}A'(x)}{4\sqrt{A(x)}} \\ A_\phi &= \frac{(m-\frac{1}{2})A'(x) + (A(x)W'(x)/W(x))'x}{\sqrt{A'^2(0) - 2A''A(0)}}. \end{aligned} \tag{3.13}$$

The 2-form of magnetic field is calculated as

$$B = \frac{(m-\frac{1}{2})A''(x) + (A(x)W'(x)/W(x))'}{\sqrt{A'^2(0) - 2A''A(0)}}\sqrt{A(x)}\,\mathrm{d}\theta\wedge\mathrm{d}\phi \tag{3.14}$$

which, as we shall see, will be related to the magnetic monopole field on the $SL(2,c)/GL(1,c)$ manifold. Thus, using equation (3.11), we can find the time-independent Dirac equation in the presence of the magnetic field (3.14):

$$D_2(m)\,\Psi_{n,l,m}(\theta,\phi) = \sqrt{E(n,m)}\,\Psi_{n,l,m}(\theta,\phi) \tag{3.15}$$

where the Dirac operator $D_2(m)$ on $SL(2,c)/GL(1,c)$ manifold is

$$D_2(m) = \begin{pmatrix} 0 & -\mathrm{i}J_-(m) \\ \mathrm{i}J_+(m) & 0 \end{pmatrix}. \tag{3.16}$$

It is clear that by acting the Dirac operator $D_2(m)$ from the left on both sides of equation (3.15) we obtain the shape invariance equations (2.18), again. Thus, it shows that the shape invariance, as discussed in the previous section, solves the Dirac equation for a charged spin-$\frac{1}{2}$ particle in this spacetime in the presence of a magnetic monopole field. The Dirac spinors $\Psi_{n,l,m}(\theta,\phi)$ on the $SL(2,c)/GL(1,c)$ manifold are expressed in terms of the master function in a compact form.

The relation (3.14) which was obtained from the theory of representation, automatically describes the Dirac quantization condition of the magnetic charge in terms of m, if $A''(x)\neq 0$, i.e. if $SL(2,c)/GL(1,c)$ is $SU(2)/U(1)$ or $SU(1,1)/U(1)$ (see the appendix). We give some of the results discussed above for some master functions $A(x)$ in the appendix, with the restriction that the variables θ and ϕ are real values. There, for instance, we obtain a Dirac equation for $A(x) = 1-x^2$ on s^2 in the presence of a magnetic monopole field with monopole harmonics as its eigenfunctions. For $A(x) = x$, the solution of the Dirac equation is related to a constant magnetic field along the vertical to the two-dimensional flat manifold.

4. Dirac spinors on the homogeneous manifold $SL(2,c)/GL(1,c)$ as an $N = 1$ chiral supersymmetry algebra representation

We define chiral supersymmetry generators as a pair of suitable fermion creation and annihilation operators [12]

$$Q_\pm(m) = \tfrac{1}{2}(I\pm\gamma^5)D_2(m) \tag{4.1}$$

with $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Now, one can easily show that

$$D_2(m) = Q_+(m) + Q_-(m). \tag{4.2}$$

It is clear that $Q_+(m)$ and $Q_-(m)$ are nilpotent, i.e. $Q_+^2(m) = Q_-^2(m) = 0$. Therefore, the square of the time-independent Dirac operator, together with $Q_+(m)$ and $Q_-(m)$, form the usual $N = 1$ supersymmetry algebra

$$D_2^2(m) = \{Q_+(m), Q_-(m)\} \qquad [Q_\pm(m), D_2^2(m)] = 0. \tag{4.3}$$

Thus, with regard to

$$\begin{aligned} Q_+(m)\,\Psi_{n,l,m}(\theta,\phi) &= \sqrt{E(n,m)}\begin{pmatrix} \psi_{n,l,m-1}(\theta,\phi,0) \\ 0 \end{pmatrix} \\ Q_-(m)\,\Psi_{n,l,m}(\theta,\phi) &= \sqrt{E(n,m)}\begin{pmatrix} 0 \\ \mathrm{i}\psi_{n,l,m}(\theta,\phi,0) \end{pmatrix} \\ D_2^2(m)\,\Psi_{n,l,m}(\theta,\phi) &= E(n,m)\,\Psi_{n,l,m}(\theta,\phi) \end{aligned} \tag{4.4}$$

we can deduce that the Dirac spinors $\Psi_{n,l,m}(\theta,\phi)$ over the homogeneous manifold $SL(2,c)/GL(1,c)$ are the bases of representation of the $N = 1$ chiral supersymmetry algebra.

5. Spinors on the homogeneous manifold $SL(2,c)/GL(1,c)$ as a representation of the shape invariance symmetry and unitary parasupersymmetry algebra

At the outset we show that the spinors introduced in section 2 form the bases for realization of shape invariance symmetry. The appropriate operators for expressing shape invariance of spinors are

$$\tilde{J}_\pm(m) = \pm\mathrm{i}\begin{pmatrix} \sqrt{\dfrac{E(n,m)}{E(n,m-1)}}J_\pm(m-1) & 0 \\ 0 & J_\pm(m) \end{pmatrix}. \tag{5.1}$$

By using equations (2.18), it is obvious that shape invariance equations of spinors on the homogeneous manifold $SL(2,c)/GL(1,c)$ are given as

$$\begin{aligned} \tilde{J}_+(m)\tilde{J}_-(m)\,\Psi_{n,l,m}(\theta,\phi) &= E(n,m)\,\Psi_{n,l,m}(\theta,\phi) \\ \tilde{J}_-(m)\tilde{J}_+(m)\,\Psi_{n,l,m-1}(\theta,\phi) &= E(n,m)\,\Psi_{n,l,m-1}(\theta,\phi) \end{aligned} \tag{5.2}$$

or

$$\begin{aligned} \tilde{J}_+(m)\,\Psi_{n,l,m-1}(\theta,\phi) &= \sqrt{E(n,m)}\,\Psi_{n,l,m}(\theta,\phi) \\ \tilde{J}_-(m)\,\Psi_{n,l,m}(\theta,\phi) &= \sqrt{E(n,m)}\,\Psi_{n,l,m-1}(\theta,\phi). \end{aligned} \tag{5.3}$$

The relations (5.3) show that spinors can be generated algebraically. Now, we show that spinors of the $SL(2,c)/GL(1,c)$ manifold form the bases of representation of the Rubakov–Spiridonov [4, 6] unitary parasuperalgebra of order p, such that $1 \leqslant p \leqslant n+1$. Let us define parafermionic generators Q_1, Q_2 and the bosonic generator H as $2(p+1)\times 2(p+1)$ block matrices:

$$\begin{aligned}(Q_1)_{mm'} &= \tilde{J}_-(m)\delta_{m+1,m'}\\ (Q_2)_{mm'} &= \tilde{J}_+(m')\delta_{m,m'+1}\\ (H)_{mm'} &= \tilde{H}_m\delta_{m,m'} \qquad m,m'=1,2,\ldots,p+1\end{aligned} \tag{5.4}$$

where $\tilde{H}_m$, similarly to $\tilde{J}_+(m)$ and $\tilde{J}_-(m)$, are 2×2 matrices. Generators Q_1, Q_2 and H, as in [3], form a unitary parasuperalgebra of order p, provided that $\tilde{J}_+(m)$, $\tilde{J}_-(m)$ and $\tilde{H}_m$ satisfy the relations

$$\begin{aligned}&\tilde{J}_+(p-1)\cdots\tilde{J}_+(2)\tilde{J}_+(1)\tilde{J}_-(1)\tilde{J}_+(1)+\cdots+\tilde{J}_+(p-1)\tilde{J}_-(p-1)\\ &\qquad\times\tilde{J}_+(p-1)\tilde{J}_+(p-2)\cdots\tilde{J}_+(1)+\tilde{J}_-(p)\tilde{J}_+(p)\tilde{J}_+(p-1)\cdots\tilde{J}_+(1)\\ &\quad=2p\tilde{J}_+(p-1)\tilde{J}_+(p-2)\cdots\tilde{J}_+(1)\tilde{H}_1\\ &\tilde{J}_+(p)\cdots\tilde{J}_+(2)\tilde{J}_+(1)\tilde{J}_-(1)+\tilde{J}_+(p)\cdots\tilde{J}_+(3)\tilde{J}_+(2)\tilde{J}_-(2)\tilde{J}_+(2)+\cdots\\ &\qquad+\tilde{J}_+(p)\tilde{J}_-(p)\tilde{J}_+(p)\tilde{J}_+(p-1)\cdots\tilde{J}_+(2)\\ &\quad=2p\tilde{J}_+(p)\tilde{J}_+(p-1)\cdots\tilde{J}_+(2)\tilde{H}_2\\ &\tilde{J}_-(1)\cdots\tilde{J}_-(p-1)\tilde{J}_-(p)\tilde{J}_+(p)+\tilde{J}_-(1)\cdots\tilde{J}_-(p-2)\tilde{J}_-(p-1)\\ &\qquad\times\tilde{J}_+(p-1)\tilde{J}_-(p-1)+\cdots+\tilde{J}_-(1)\tilde{J}_+(1)\tilde{J}_-(1)\tilde{J}_-(2)\cdots\tilde{J}_-(p-1)\\ &\quad=2p\tilde{J}_-(1)\tilde{J}_-(2)\cdots\tilde{J}_-(p-1)\tilde{H}_p\\ &\tilde{J}_-(2)\cdots\tilde{J}_-(p-1)\tilde{J}_-(p)\tilde{J}_+(p)\tilde{J}_-(p)+\cdots+\tilde{J}_-(2)\tilde{J}_+(2)\tilde{J}_-(2)\tilde{J}_-(3)\cdots\tilde{J}_-(p)\\ &\qquad+\tilde{J}_+(1)\tilde{J}_-(1)\tilde{J}_-(2)\cdots\tilde{J}_-(p)=2p\tilde{J}_-(2)\tilde{J}_-(3)\cdots\tilde{J}_-(p)\tilde{H}_{p+1}\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\tilde{H}_m\tilde{J}_-(m)&=\tilde{J}_-(m)\tilde{H}_{m+1}\\ \tilde{H}_{m+1}\tilde{J}_+(m)&=\tilde{J}_+(m)\tilde{H}_m.\end{aligned} \tag{5.6}$$

Similarly to [3], one can prove that the $\tilde{H}_m$ satisfying equations (5.5) and (5.6) are

$$\begin{aligned}\tilde{H}_m&=\frac{1}{2}\begin{pmatrix}\dfrac{E(n,m)}{E(n,m-1)}J_-(m-1)J_+(m-1) & 0\\ 0 & J_-(m)J_+(m)\end{pmatrix}\\ &\qquad+\frac{1}{2}\left\{\frac{p-2m+1}{2}\left(\frac{A(x)W'(x)}{W(x)}\right)'+\frac{p^2-3m^2+3m-1}{6}A''(x)\right\}I\\ &\qquad m=1,2,\ldots,p\\ \tilde{H}_{p+1}&=\frac{1}{2}\begin{pmatrix}\dfrac{E(n,p)}{E(n,p-1)}J_+(p-1)J_-(p-1) & 0\\ 0 & J_+(p)J_-(p)\end{pmatrix}\\ &\qquad+\frac{1}{2}\left\{\frac{1-p}{2}\left(\frac{A(x)W'(x)}{W(x)}\right)'+\frac{-2p^2+3p-1}{6}A''(x)\right\}I\end{aligned} \tag{5.7}$$

and that all $\tilde{H}_m$ are isospectral:

$$\tilde{E}=\frac{p-2n-1}{4}\left(\frac{A(x)W'(x)}{W(x)}\right)'+\frac{p^2-3n^2-3n-1}{12}A''(x). \tag{5.8}$$

With regard to

$$\tilde{H}_m \Psi_{n,l,m-1}(\theta,\phi) = \tilde{E}\Psi_{n,l,m-1}(\theta,\phi) \qquad m = 1, 2, \ldots, p+1 \tag{5.9}$$

let us define $\Omega(\theta,\phi)$ as a column matrix with $2(p+1)$ rows

$$(\Omega(\theta,\phi))_m := \Psi_{n,l,m}(\theta,\phi) \qquad m = 0, 1, \ldots, p \tag{5.10}$$

then we have

$$H\Omega(\theta,\phi) = \tilde{E}\Omega(\theta,\phi). \tag{5.11}$$

Therefore, the spinors $\Psi_{n,l,m}(\theta,\phi)$ form the bases for the representation of a unitary parasupersymmetry algebra of order p on the homogeneous manifold $SL(2,c)/GL(1,c)$.

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Appendix

$A(x) = 1 - x^2$

$W(x) = (1-x)^{\alpha}(1+x)^{\alpha-2l} \qquad -1 \leqslant x \leqslant +1 \qquad \alpha > -1 \qquad l < \frac{1}{2}(1+\alpha)$

$x = -\cos\theta \qquad 0 \leqslant \theta < 2\pi$

$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\theta^2 - \sin^2\theta\,\mathrm{d}\phi^2 \qquad \dfrac{SL(2,c)}{GL(1,c)} = \dfrac{SU(2)}{U(1)}$

$J_+(m) = \dfrac{\partial}{\partial\theta} - \dfrac{\mathrm{i}}{\sin\theta}\dfrac{\partial}{\partial\phi} - \dfrac{2(m-l+\alpha)-1}{2\tan\theta}$

$J_-(m) = -\dfrac{\partial}{\partial\theta} - \dfrac{\mathrm{i}}{\sin\theta}\dfrac{\partial}{\partial\phi} - \dfrac{2(m-l+\alpha)-1}{2\tan\theta}$

$B = -\left(m - l + \alpha - \frac{1}{2}\right)\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi$

$E(n,m) = (n-m+1)(n+m-2l+2\alpha)$

$$\psi_{n.l,m}(\theta,\phi,0) = a_n(-1)^m \mathrm{e}^{\mathrm{i}l\phi}\Bigg[(1-x)^{-(2m+2\alpha-1)/4}(1+x)^{-(2m-4l+2\alpha-1)/4} \times\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}\left((1-x)^{n+\alpha}(1+x)^{n-2l+\alpha}\right)\Bigg]_{x=-\cos\theta}.$$

$A(x) = x^2 - 1$

$W(x) = (x-1)^{\alpha}(x+1)^{\alpha+2l} \qquad +1 \leqslant x < +\infty \qquad \alpha > -1 \qquad l < -\frac{1}{2}(1+\alpha)$

$x = \cosh\theta \qquad 0 \leqslant \theta < +\infty$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\theta^2 - \sinh^2\theta\,\mathrm{d}\phi^2 \qquad \frac{SL(2,c)}{GL(1,c)} = \frac{SU(1,1)}{U(1)}$$

$$J_+(m) = \frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sinh\theta}\frac{\partial}{\partial\phi} - \frac{2(m+l+\alpha)-1}{2\tanh\theta}$$

$$J_-(m) = -\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sinh\theta}\frac{\partial}{\partial\phi} - \frac{2(m+l+\alpha)-1}{2\tanh\theta}$$

$$B = (m+l+\alpha-\tfrac{1}{2})\sinh\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$$

$$E(n,m) = -(n-m+1)(n+m+2l+2\alpha)$$

$$\psi_{n.l,m}(\theta,\phi,0) = a_n(-1)^m \mathrm{e}^{\mathrm{i}l\phi}\Bigg[(x-1)^{-(2m+2\alpha-1)/4}(x+1)^{-(2m+4l+2\alpha-1)/4} \times\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}\big((x-1)^{n+\alpha}(x+1)^{n+2l+\alpha}\big)\Bigg]_{x=\cosh\theta}.$$

$$A(x) = x(1-x)$$

$$W(x) = x^{-2l}(1-x)^{\beta} \qquad 0 \leqslant x \leqslant +1 \qquad \beta > -1 \qquad l < \tfrac{1}{2}$$

$$x = \tfrac{1}{2}(1-\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\theta^2 - \sin^2\theta\,\mathrm{d}\phi^2 \qquad \frac{SL(2,c)}{GL(1,c)} = \frac{SU(2)}{U(1)}$$

$$J_+(m) = \frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{2(m-l)+\beta-1}{2\tan\theta} - \frac{2l-\beta}{2\sin\theta}$$

$$J_-(m) = -\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{2(m-l)+\beta-1}{2\tan\theta} - \frac{2l-\beta}{2\sin\theta}$$

$$B = -\big(m-l+\tfrac{1}{2}\beta-\tfrac{1}{2}\big)\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$$

$$E(n,m) = (n-m+1)(n+m-2l+\beta)$$

$$\psi_{n.l,m}(\theta,\phi,0) = a_n(-1)^m \mathrm{e}^{2\mathrm{i}l\phi}\Bigg[x^{-(2m-4l-1)/4}(1-x)^{-(2m+2\beta-1)/4} \times\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}\big(x^{n-2l}(1-x)^{n+\beta}\big)\Bigg]_{x=(1-\cos\theta)/2}.$$

$$A(x) = x(1+x)$$

$$W(x) = x^{-2l}(1+x)^{\beta} \qquad 0 \leqslant x < +\infty \qquad \beta > -1 \qquad l < \tfrac{1}{2}$$

$$x = \tfrac{1}{2}(\cosh\theta - 1) \qquad 0 \leqslant \theta < \infty$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\theta^2 - \sinh^2\theta\,\mathrm{d}\phi^2 \qquad \frac{SL(2,c)}{GL(1,c)} = \frac{SU(1,1)}{U(1)}$$

$$J_+(m) = \frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sinh\theta}\frac{\partial}{\partial\phi} - \frac{2(m-l)+\beta-1}{2\tanh\theta} - \frac{2l-\beta}{2\sinh\theta}$$

$$J_-(m) = -\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sinh\theta}\frac{\partial}{\partial\phi} - \frac{2(m-l)+\beta-1}{2\tanh\theta} - \frac{2l-\beta}{2\sinh\theta}$$

$$B = (m - l + \tfrac{1}{2}\beta - \tfrac{1}{2})\sinh\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$$

$$E(n,m) = -(n-m+1)(n+m-2l+\beta)$$

$$\psi_{n.l,m}(\theta,\phi,0) = a_n(-1)^m \mathrm{e}^{2\mathrm{i}l\phi}\Bigg[x^{-(2m-4l-1)/4}(1+x)^{-(2m+2\beta-1)/4} \times\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}\left(x^{n-2l}(1+x)^{n+\beta}\right)\Bigg]_{x=(\cosh\theta-1)/2}.$$

$$A(x) = x$$

$$W(x) = x^{-2l}\mathrm{e}^{-\beta x} \qquad 0 \leqslant x < +\infty \qquad \beta > 0 \qquad l < \tfrac{1}{2}$$

$$x = \tfrac{1}{4}\theta^2 \qquad 0 \leqslant \theta < +\infty$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\theta^2 - \theta^2\,\mathrm{d}\phi^2 \qquad \frac{SL(2,c)}{GL(1,c)} = \frac{H_4}{U(1)\otimes U(1)}$$

$$J_+(m) = \frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\theta}\frac{\partial}{\partial\phi} - \frac{2m-1}{2\theta} + \frac{\beta}{4}\theta$$

$$J_-(m) = -\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\theta}\frac{\partial}{\partial\phi} - \frac{2m-1}{2\theta} + \frac{\beta}{4}\theta$$

$$B = -\tfrac{1}{2}\beta\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$$

$$E(n,m) = \beta(n-m+1)$$

$$\psi_{n.l,m}(\theta,\phi,0) = a_n(-1)^m \mathrm{e}^{2\mathrm{i}l\phi}\left[x^{-(2m-4l-1)/4}\mathrm{e}^{\frac{1}{2}\beta x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}\left(x^{n-2l}\mathrm{e}^{-\beta x}\right)\right]_{x=\frac{1}{4}\theta^2}$$

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